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A note on the Lax pairs for Painlevé equations

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Abstract. For the classical Painlevé equations, besides the method of similarity reduction of Lax pairs for integrable partial differential equations, two ways are known for Lax pair generation. The first is based on the confluence procedure in Fuchs' linear ODE with four regular singularities isomonodromy deformation which is governed by the sixth Painlevé equation. The second method treats the hypergeometric equation and confluent hypergeometric equations as the isomonodromy deformation equations for the triangular systems of ODEs, in whose non-triangular extensions give rise to the Lax pairs for the Painlevé equations.

The theory of integrable integral operators suggests a new way of Lax pair generation for the classical Painlevé equations. This method involves a special kind of gauge transformation that is applied to linear systems which are exactly solvable in terms of the classical special functions. Some of the Lax pairs we introduce are known, others are new. The question of gauge equivalence of different Lax pairs for the Painlevé equations is considered as well.

1. Introduction

The Painlevé equations [1–3], originally found as the only irreducible second-order ordinary differential equations (ODEs) $y_{xx} = R(x, y, y_x)$ rational in y_x, algebraic in y and analytic in x such that their general solutions have no movable branch points and essential singularities, are known to govern the so-called isomonodromy deformations of certain linear differential equations with rational coefficients. In other words, the Painlevé equations can be written in the form of compatibility conditions for some over-determined linear systems of (2×2) matrix equations which are now called the Lax pairs.

The first such system was found by Fuchs [4] who investigated the isomonodromy deformations of the linear Fuchsian second-order scalar ODE with four singular points. Fuchs revealed that preserving the monodromy properties of such a linear equation, with necessity, leads to deformations of its coefficients in accord with the sixth Painlevé equation (PVI). Observations that other Painlevé equations can be obtained by the use of certain scaling limits from PVI [5, 3] gave rise to consideration in the articles of Garnier and Boutroux [6, 7] of the confluences of singular points in the Fuchs' linear equation for PVI and to the pairs of linear equations for all the Painlevé equations [6] written in the equivalent matrix form by Jimbo and Miwa [8]. Below, we call the pairs the Garnier systems, or the Garnier pairs.

Another way of generating Lax pairs for the Painlevé equations was introduced in [9]. The authors have shown that the Airy, Bessel and parabolic cylinder equations [10] can be written as the compatibility conditions for triangular (2×2) matrix systems. Non-triangular generalizations of these Lax pairs yield the Lax pairs for the Painlevé equations of the second, third and fourth kind, respectively.

In the present paper, we explain a new scheme of generating Lax pairs of the classical Painlevé equations which allows us to elucidate another relationship between the linear and the nonlinear special functions, i.e. between (confluent) hypergeometric functions and Painlevé functions. The idea of this scheme comes from the fact that there exists a close relationship between the integrable integral operators, the Riemann–Hilbert problem and isomonodromy deformations of the linear equations with rational coefficients, discovered by Its, Izergin, Korepin and Slavnov [11].

Consider an integral operator in the proper functional class on the set of finite intervals $\Gamma = \cup_{j=0}^{n}(a_j; a_{j+1})$,

$$(Ky)(\lambda) = \int_\Gamma K(\lambda, \mu) y(\mu)\, \mathrm{d}\mu \qquad \lambda, \mu \in \Gamma \subset \mathbb{C} \tag{1}$$

with the scalar kernel of the form introduced by Troy and Widom in [12] (for more general situations see [13, 14])

$$K(\lambda, \mu) = \frac{\varphi(\lambda)\psi(\mu) - \varphi(\mu)\psi(\lambda)}{\lambda - \mu} \tag{2}$$

where the vectors

$$f(\lambda) = \begin{pmatrix} \psi(\lambda) \\ \varphi(\lambda) \end{pmatrix} \qquad g(\mu) = \begin{pmatrix} -\varphi(\mu) \\ \psi(\mu) \end{pmatrix} \tag{3}$$

are related to the linear differential equation

$$\frac{\mathrm{d}f}{\mathrm{d}\lambda} = A_0(\lambda) f \qquad A_0 = \begin{pmatrix} a_3 & a_+ \\ a_- & -a_3 \end{pmatrix} \tag{4}$$

with A_0 rationally dependent on λ.

The first example of such kernels, introduced in [15] for a description of the quantum correlation functions and in [16] for a description of the classical level spacing distribution function in the ‘bulk’ of a spectrum of the Gaussian unitary ensemble (GUE), is the famous sine kernel

$$K(\lambda, \mu) = \frac{\sin \pi(\lambda - \mu)}{\pi(\lambda - \mu)} \tag{5}$$

with the exponent functions $\varphi(\lambda)$ and $\psi(\lambda)$,

$$\varphi(\lambda) = \frac{\mathrm{e}^{\mathrm{i}\pi\lambda}}{2\pi\mathrm{i}} \qquad \psi(\lambda) = \frac{\mathrm{e}^{-\mathrm{i}\pi\lambda}}{2\pi\mathrm{i}}.$$

The corresponding linear matrix (equation (4)) is a constant digital matrix,

$$A_0(\lambda) = \begin{pmatrix} \mathrm{i}\pi & 0 \\ 0 & -\mathrm{i}\pi \end{pmatrix}.$$

Surprisingly, the Fredholm determinant of the integral operator with kernel (5) on the segment $(-x; x)$, $\det(I - K\chi_{(0;x)})$, where we denote by χ_Γ the characteristic function of the set Γ, is related to the classical fifth Painlevé equation (PV).

The next examples of such a kind are presented in [17, 18]. The first is the Airy kernel

$$K(\lambda, \mu) = \frac{Ai(\lambda)Ai'(\mu) - Ai'(\lambda)Ai(\mu)}{\lambda - \mu} \tag{6}$$

where $Ai(\lambda)$ and $Ai'(\lambda)$ are the classical Airy function and its derivative [10], which describe the level spacing distribution function for the GUE at the 'edge' of a spectrum. The second is the Bessel kernel

$$K(\lambda,\mu) = \frac{J_\alpha(\sqrt{\lambda})\sqrt{\mu}J'_\alpha(\sqrt{\mu}) - \sqrt{\lambda}J'_\alpha(\sqrt{\lambda})J_\alpha(\sqrt{\mu})}{2(\lambda-\mu)} \tag{7}$$

for the 'hard edge' of this spectrum at $\lambda = 0$. In the Airy kernel case, in our notation, the functions

$$\varphi(\lambda) = Ai(\lambda) \qquad \psi(\lambda) = Ai'(\lambda)$$

with the Airy function $Ai(\lambda)$ form the vector solution of (4), with the matrix $A_0(\lambda)$ given by

$$A_0(\lambda) = \begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix}. \tag{8}$$

In the Bessel kernel case, the functions

$$\varphi(\lambda) = J_\alpha(\sqrt{\lambda})/2 \qquad \psi(\lambda) = \sqrt{\lambda}J'_\alpha(\sqrt{\lambda})$$

with the Bessel function $J_\alpha(x)$ yield the vector solution of (4) with

$$A_0(\lambda) = \begin{pmatrix} 0 & 1/4\lambda \\ -1 + (\alpha^2/\lambda) & 0 \end{pmatrix}. \tag{9}$$

As shown in [17, 18], Fredholm determinants of such operators on the half-infinite interval $(x, +\infty)$ are related to the second and third Painlevé equations (PII, PIII) respectively.

Some kernels, 'beyond Airy', introduced in [12] are expressed in terms of Hermite, Laguerre and Jacobi polynomials and related to the fourth, fifth and sixth Painlevé equations (PIV, PV and PVI, respectively).

Following [12, 13], let us introduce the functions F and G as the solutions of the integral equations $F = \varphi + KF$, $G = \psi + KG$, i.e.

$$F = (1-K)^{-1}\varphi \qquad G = (I-K)^{-1}\psi \tag{10}$$

and the matrix function $Y(\lambda)$ by the use of the Cauchy integral

$$Y(\lambda) = I - \int_\Gamma \frac{F(\mu)g^{\mathrm{T}}(\mu)}{\mu-\lambda}\,\mathrm{d}\mu. \tag{11}$$

It follows from the properties of the Cauchy integrals (see, e.g. [19]) that this function is holomorphic outside Γ, normalized to the unit at infinity, may have logarithmic or integrable algebraic singularities at the points $a_j \in \partial\Gamma$ and is characterized by the jump on Γ

$$Y_+(\lambda) = Y_-(\lambda)H(\lambda) \qquad H(\lambda) = I - 2\pi\mathrm{i}\begin{pmatrix} -\psi(\lambda)\varphi(\lambda) & \psi^2(\lambda) \\ -\varphi^2(\lambda) & \psi(\lambda)\varphi(\lambda) \end{pmatrix} \qquad \lambda \in \Gamma. \tag{12}$$

The special form of the jump immediately leads to a very specific gauge transformation of the matrix (2×2) equation

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\lambda} = A_0(\lambda)\Phi. \tag{13}$$

Indeed, let $\Phi(\lambda)$ be a fundamental solution of (13), i.e. $\det\Phi(\lambda) = 1$. Then the vector $f(\lambda) = (\psi(\lambda);\varphi(\lambda))^{\mathrm{T}}$ is given by the product

$$f(\lambda) = \Phi(\lambda)p \qquad p = \begin{pmatrix} q \\ p \end{pmatrix} = \text{ constant.} \tag{14}$$

At the same time, the vector $g(\lambda) = (-\varphi(\lambda);\psi(\lambda))^{\mathrm{T}}$ is given by the product

$$g(\lambda) = (\Phi(\lambda)^{-1})^{\mathrm{T}}q \qquad q = \begin{pmatrix} -p \\ q \end{pmatrix} = \text{ constant.} \tag{15}$$

Therefore, the product $\Phi^{-1}(\lambda)H(\lambda)\Phi(\lambda)$ does not depend on λ

$$\Phi^{-1}(\lambda)H(\lambda)\Phi(\lambda) = I - 2\pi\mathrm{i}\begin{pmatrix} -pq & q^2 \\ -p^2 & pq \end{pmatrix} = \text{ constant.} \tag{16}$$

Thus the new (2×2) matrix function

$$\Psi(\lambda) = Y(\lambda)\Phi(\lambda) \tag{17}$$

has constant jump on Γ and satisfies a linear differential equation [14]

$$\frac{\mathrm{d}\Psi}{\mathrm{d}\lambda} = A(\lambda)\Psi \tag{18}$$

with the matrix A, rational in λ, related to the 'vacuum' matrix A_0 by the equation

$$A = Y A_0 Y^{-1} + Y_\lambda Y^{-1}. \tag{19}$$

The above-mentioned Riemann–Hilbert problem for $\Psi(\lambda)$ (for details, see [14]) contains natural isomonodromy deformation parameters $a_j \in \partial\Gamma$, so that besides the equation in λ (18)

$$\frac{\partial\Psi}{\partial\lambda} = A(\lambda)\Psi \equiv \left(\tilde{A}_0(\lambda) + \sum_j \frac{A_j}{\lambda - a_j}\right)\Psi \tag{20}$$

where A_j are constant matrices and $\tilde{A}_0(\lambda)$ is a matrix, rational in λ, with no singularity at a_j, the function $\Psi(\lambda)$ satisfies

$$\mathrm{d}\Psi = U(\lambda)\Psi \equiv \left(-\sum_j \frac{A_j}{\lambda - a_j}\,\mathrm{d}a_j\right)\Psi. \tag{21}$$

The compatibility condition for the system (20) and (21),

$$\mathrm{d}A - \frac{\partial U}{\partial\lambda} + [A, U] = 0 \tag{22}$$

which holds identically in λ, gives rise to the system of nonlinear differential equations called the system of isomonodromy deformations [8].

Below, we show that, in the simplest situations, such systems reduce to second-order ODEs that are equivalent to some of the classical Painlevé equations. The investigation described in this paper was essentially simplified by the use of symbolic computations based on simple manipulations on *Maple V*. The reductions of the compatibility condition systems (equation (22)) to Painlevé equations are computed according to an algorithm in differential algebra as presented in [20]. In practice we used the *Maple V.5.1* package *diffalg*, which is the implementation of such an algorithm. Information about this package can be found on the web page `http;//daisy.uwaterloo.ca/~ehubert/Diffalg`.

Actually, the Lax pair for the 34th Painlevé equation (P34) from the list of Painlevé and Gambier [3] that is related to the Airy kernel was found 'by hands' in [14] and is the gauge equivalent to the pair of Flaschka and Newell [21] for PII. The new Lax pair for PIII is found by use of symbolic computation. The Lax pair for PIV, related to the integral operator with the Weber–Hermite kernel of the first kind (41) and (42), coincides up to a diagonal gauge transformation with the pair of Garnier [6, 8]. The Weber–Hermite kernel of the second kind (54) and (55) leads to another Lax pair for PIV evaluated by the use of symbolic computations. We show that this Lax pair, as well as the pair for PIV found in [23], are both gauge equivalent to the pair of Garnier. Finally, we observe that the Garnier systems [6, 8] for PV and PVI are related in the very same way to the Whittaker and hypergeometric kernels, respectively.

2. Airy kernel and PII

The simplest non-trivial example of the Lax pairs related to the integral operators (1) with the Tracy–Widom kernel (2) and (3) is the pair coming from consideration of the operator with the Airy kernel (6) on the ray $(x;\infty)$. This example is found and investigated in full detail in [14]. Here, we give an overview.

The 'vacuum' matrix, equation (13), has the form of (8), and because the gauge matrix $Y(\lambda)$ tends to I at infinity and the contour Γ, in this case, consists of the ray $(x;\infty)$, the perturbed matrix equation is characterized by the matrix (19) of the Jordan form in the main order at infinity, with the only regular singularity at $\lambda = x$:

$$\Psi_\lambda \Psi^{-1} = A = \begin{pmatrix} a & \lambda + b \\ 1 & -a \end{pmatrix} + \frac{1}{\lambda - x}\begin{pmatrix} p & q \\ r & -p \end{pmatrix} \tag{23}$$

$$\Psi_x \Psi^{-1} = U = -\frac{1}{\lambda - x}\begin{pmatrix} p & q \\ r & -p \end{pmatrix}. \tag{24}$$

The compatibility condition of (23) and (24) is equivalent to the system of equations

$$\begin{aligned} &b = -r - a^2 \\ &p = ar + \epsilon\sqrt{-r^3 + xr^2 - ar + \nu^2} \qquad \epsilon^2 = 1 \\ &q = r^2 - a^2 r - xr + a - 2a\epsilon\sqrt{-r^3 + xr^2 - ar + \nu^2} \\ &a_x = r \\ &r_x = 2\epsilon\sqrt{-r^3 + xr^2 - ar + \nu^2} \qquad \nu = \text{ constant.} \end{aligned} \tag{25}$$

The last two equations of system (25) yield the second-order ODE for the function r

$$r_{xx} = \frac{(r_x)^2}{2r} - 4r^2 + 2xr - \frac{2\nu^2}{r} \tag{26}$$

which up to scaling change of r and x coincides with the classical P34 equation from the list of Painlevé–Gambier [13].

The Lax pair (23) and (24) is gauge equivalent to the Lax pair of Flaschka and Newell [21]

$$\frac{\partial \Psi^{\mathrm{FN}}}{\partial \xi} = \mathcal{A}\Psi^{\mathrm{FN}} \qquad \frac{\partial \Psi^{\mathrm{FN}}}{\partial t} = \mathcal{U}\Psi^{\mathrm{FN}} \tag{27}$$

with the matrix coefficients $\mathcal{A}$ and $\mathcal{U}$ defined by

$$\mathcal{A} = \begin{pmatrix} -\mathrm{i}(4\xi^2 + t + 2y^2) & \mathrm{i}(4y\xi + (\alpha/\xi)) - 2z \\ -\mathrm{i}(4y\xi + (\alpha/\xi)) - 2z & \mathrm{i}(4\xi^2 + t + 2y^2) \end{pmatrix} \qquad \mathcal{U} = \begin{pmatrix} -\mathrm{i}\xi & \mathrm{i}y \\ -\mathrm{i}y & \mathrm{i}\xi \end{pmatrix}. \tag{28}$$

The compatibility condition of (27) and (28),

$$y_t = z \qquad z_t = 2y^3 + ty - \alpha \qquad \alpha = \text{ constant} \tag{29}$$

is equivalent to PII

$$y_{tt} = 2y^3 + ty - \alpha \qquad \alpha = \text{ constant.} \tag{30}$$

The gauge equivalence of the Lax pairs (23) and (24), and (27) and (28) is given by

$$\begin{aligned} &\Psi(\lambda, x) = 2^{\sigma_3/6}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\xi^{\sigma_3/2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix}\Psi^{\mathrm{FN}}(\xi, t) \\ &\lambda = -2^{2/3}\xi^2 - 2^{-1/3}t \qquad x = -2^{-1/3}t \end{aligned} \tag{31}$$

where the function $u(t)$ satisfies the equation

$$u_t + \frac{1}{2}\left(z - y^2 - \frac{t}{2}\right) = 0. \tag{32}$$

The function $\Psi(\lambda, x)$ defined by (31) solves the system (23) and (24) with the coefficients

$$\begin{aligned} &a = 2^{1/3}(y+u) \qquad b = -2^{2/3}(y+u)^2 + 2^{-1/3}\left(z + y^2 + \frac{t}{2}\right) \\ &p = -\frac{\alpha}{2} + \frac{1}{4} - u\left(z + y^2 + \frac{t}{2}\right) \qquad q = 2^{1/3}u\left[\alpha - \frac{1}{2} + u\left(z + y^2 + \frac{t}{2}\right)\right] \\ &r = -2^{-1/3}\left(z + y^2 + \frac{t}{2}\right). \end{aligned} \tag{33}$$

Inversion of the gauge transformation (31) is given by the formula

$$\begin{aligned} &\Psi^{\mathrm{FN}}(\xi, t) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \mathrm{e}^{-\mathrm{i}(\pi/4)\sigma_3}(\lambda - x)^{-\sigma_3/4}\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}\Psi(\lambda, x) \\ &\xi = \mathrm{e}^{\mathrm{i}\pi/2}2^{-1/3}(\lambda - x)^{1/2} \qquad t = -2^{1/3}x \end{aligned} \tag{34}$$

where the function $v(t)$ is defined by the equation

$$v = -\frac{p \pm \nu}{r} \equiv -2^{1/3}\mu. \tag{35}$$

The function Ψ^{FN} defined by (34) solves the system (27) and (28) with the parameters

$$\begin{aligned} &y = 2^{-1/3}(a + v) \quad \alpha = \tfrac{1}{2} \pm 2\nu \\ &z = 2^{-2/3}x - 2^{1/3}r - 2^{-2/3}(a + v)^2. \end{aligned} \tag{36}$$

The equations (33) and (36) establish the known Bäcklund transformation between PII and P34 [3].

Note that the λ-equation in the Garnier system for PII found in [6, 8] has the only irregular singularity at infinity. This fact suggests there is no algebraic gauge transformation between the Lax pairs of Flaschka and Newell [21] and Garnier [6, 8].

3. Bessel kernel and PIII

Let us consider the integral operator on $\Gamma = (x; \infty)$ with kernel (7) related to the 'vacuum' equation (equation (13)) with the matrix A_0 given by (9). Since A_0 is a Jordan matrix at infinity and the matrix of gauge transformation $Y(\lambda)$ behaves like I there, the matrix A, coming from the matrix representation for the Bessel equation, is also Jordan at infinity and has the only singularity at $\lambda = x$ (for our convenience, we have used before some constant transformation to make the Jordan block of the standard form):

$$\Psi_\lambda\Psi^{-1} = A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \frac{1}{\lambda}\begin{pmatrix} a & b \\ c & -a \end{pmatrix} + \frac{1}{\lambda - x}\begin{pmatrix} p & q \\ r & -p \end{pmatrix} \tag{37}$$

$$\Psi_x\Psi^{-1} = U = -\frac{1}{\lambda - x}\begin{pmatrix} p & q \\ r & -p \end{pmatrix}. \tag{38}$$

The compatibility condition for the system (37) and (38) is equivalent to the system of equations

$$\begin{aligned} &b = (a^2 - \mu^2)\frac{y - 1}{\rho} \qquad c = -\frac{\rho}{y - 1} \\ &p = \frac{x}{2}\frac{y_x}{y - 1} - ay \qquad q = -(p^2 - \nu^2)\frac{y - 1}{\rho y} \end{aligned}$$

$$
\begin{aligned}
& r=\frac{\rho y}{y-1} \\
& a_x+a\frac{y_x}{y-1}=\frac{x}{4}\frac{(y_x)^2}{y(y-1)^2}+\frac{1}{x}\left(\mu^2 y-\frac{\nu^2}{y}\right) \\
& \rho,\mu,\nu=\text{ constant}
\end{aligned}
\tag{39}
$$

where the function y satisfies the second-order ODE

$$
y_{xx}=\left(\frac{1}{2y}+\frac{1}{y-1}\right)y_x^2-\frac{y_x}{x}+\frac{2(y-1)^2}{x^2}\left(\mu^2 y-\frac{\nu^2}{y}\right)+\frac{2\rho}{x}y. \tag{40}
$$

Equation (40) is the special case of PV [1] with the parameters

$$
\alpha=2\mu^2 \qquad \beta=-2\nu^2 \qquad \gamma=2\rho \qquad \delta=0.
$$

It is well known [22] that equation (40) is equivalent to the general PIII equation.

Since the known Lax pair for the PIII equation [6, 8] has two irregular singularities at zero and infinity, there is no algebraic gauge transformation between the Garnier system for PIII and (37) and (38).

4. Lax pairs for PIV

In this section we describe several Lax pairs that come from applying the gauge transformations (17) and (19) to the matrix equations that are exactly soluble in terms of parabolic cylinder functions and show their algebraic gauge equivalence to the Garnier system.

4.1. Weber–Hermite kernel and PIV. Case 1

Let us consider the integral operator (1) with the kernel (2) on the ray $\Gamma=(x;\infty)$ where $\varphi(\lambda)$, $\psi(\lambda)$ are given by the parabolic cylinder functions [10]

$$
\varphi(\lambda)=w_\nu(\sqrt{2}\lambda) \qquad \psi(\lambda)=w_{\nu-1}(\sqrt{2}\lambda) \qquad \frac{\mathrm{d}^2 w_\nu(z)}{\mathrm{d}z^2}=\left(\frac{z^2}{4}-\nu-\frac{1}{2}\right)w_\nu(z). \tag{41}
$$

For an integer $\nu=N$ and polynomial $w_N(z)\,\mathrm{e}^{z^2/2}$, such a kernel was introduced in [12].

Because $w_{\nu-1}(z)=(\sqrt{2}/v)(w_\nu'(z)+(z/2)w_\nu(z))$, where $v\neq 0$ does not depend on z, the vector $f(\lambda)=(\psi(\lambda);\varphi(\lambda))^{\mathrm{T}}$ solves equation (4) with the matrix

$$
A_0(\lambda)=\begin{pmatrix}\lambda & u\\ v & -\lambda\end{pmatrix} \qquad u=-\frac{2\nu}{v}. \tag{42}
$$

The gauge transformation (17) yields the Lax pair

$$
\Psi_\lambda\Psi^{-1}=A(\lambda,x)=\begin{pmatrix}\lambda+a & b\\ c & -\lambda-a\end{pmatrix}+\frac{1}{\lambda-x}\begin{pmatrix}p & q\\ r & -p\end{pmatrix} \tag{43}
$$

$$
\Psi_x\Psi^{-1}=U(\lambda,x)=-\frac{1}{\lambda-x}\begin{pmatrix}p & q\\ r & -p\end{pmatrix} \tag{44}
$$

which is compatible if

$$
\begin{cases}
a_x=0\\
b_x=2q\\
c_x=-2r\\
p_x=br-cq\\
q_x=2(x+a)q-2bp\\
r_x=-2(x+a)r+2cp.
\end{cases} \tag{45}
$$

This system allows the integrals

$$\begin{cases} a = a_0 = \text{ constant} \\ a^2 + bc + 2p = \rho = \text{ constant} \\ p^2 + qr = \gamma^2 = \text{ constant.} \end{cases} \tag{46}$$

Moreover, without loss of generality, we put

$$a = a_0 = 0 \tag{47}$$

that is equivalent to a simultaneous shift of λ and x in the constant a_0. As to other integrals in (46), the parameter ρ describes the formal monodromy of the function Ψ at infinity, while γ has the same sense for the singular point $\lambda = x$. Using the integrals, we exclude p,

$$p = \frac{\rho - bc}{2} \tag{48}$$

and one of the functions r or q,

$$r = \frac{1}{4q}(4\gamma^2 - (bc - \rho)^2) \qquad \text{or} \qquad q = \frac{1}{4r}(4\gamma^2 - (bc - \rho)^2) \tag{49}$$

to reduce the system (45) to one for the functions b, c and q (or r):

$$\begin{cases} b_x = 2q, \\ c_x = -\dfrac{1}{q}\left(2\gamma^2 - \dfrac{1}{2}(bc - \rho)^2\right) \\ q_x = 2xq + b(bc - \rho) \end{cases} \qquad \text{or} \qquad \begin{cases} b_x = \dfrac{1}{r}\left(2\gamma^2 - \dfrac{1}{2}(bc - \rho)^2\right) \\ c_x = -2r \\ r_x = -2xr - c(bc - \rho). \end{cases} \tag{50}$$

Elementary computation shows that the functions $Q = -2q/b$ and $R = -2r/c$ satisfy the classical PIV equation [3]

$$Q = -\frac{2q}{b} : \qquad Q_{xx} = \frac{Q_x^2}{2Q} + \frac{3}{2}Q^3 + 4xQ^2 + 2(x^2 + \rho + 1)Q - \frac{8\gamma^2}{Q} \tag{51}$$

with the parameters

$$a = -\rho - 1 \qquad \beta = 16\gamma^2.$$

Moreover, b satisfies the first-order, linear ODE and becomes the so-called auxiliary function

$$\frac{b_x}{b} = -Q \tag{52}$$

while all other coefficients are determined uniquely:

$$c = \frac{1}{b}\left(-\frac{1}{2}Q_x + \frac{1}{2}Q^2 + xQ + \rho\right) \qquad p = -\frac{1}{2}(bc - \rho)$$

$$q = -\frac{1}{2}bQ \qquad r = \frac{1}{4q}(4\gamma^2 - (bc - \rho)^2).$$

Similarly,

$$R = -\frac{2r}{c} : \qquad R_{xx} = \frac{R_x^2}{2R} + \frac{3}{2}R^3 + 4xR^2 + 2(x^2 + \rho - 1)R - \frac{8\gamma^2}{R} \tag{53}$$

which coincides with PIV [3] with the parameters

$$\alpha = -\rho + 1 \qquad \beta = 16\gamma^2$$

while

$$\frac{c_x}{c} = R \qquad b = \frac{1}{c}\left(\frac{1}{2}R_x + \frac{1}{2}R^2 + xR + \rho\right) \qquad p = -\frac{1}{2}(bc - \rho)$$

$$r = -\frac{1}{2}cR \qquad q = \frac{1}{4r}(4\gamma^2 - (bc - \rho)^2).$$

4.2. Weber–Hermite kernel and PIV. Case 2

Another kind of integral operator on the ray $\Gamma = (x; \infty)$ with the kernel (2) is determined by the functions $\varphi(\lambda)$, $\psi(\lambda)$ given by the parabolic cylinder function and its derivative:

$$\varphi(\lambda) = w_\nu(\sqrt{2}\lambda) \qquad \psi(\lambda) = \sqrt{2}w'_\nu(\sqrt{2}\lambda) \qquad \frac{\mathrm{d}^2 w_\nu(z)}{\mathrm{d}z^2} = \left(\frac{z^2}{4} - \nu - \frac{1}{2}\right) w_\nu(z). \tag{54}$$

Now, the vector $f(\lambda) = (\psi(\lambda), \varphi(\lambda))^{\mathrm{T}}$ solves (4) with the matrix

$$A_0(\lambda) = \begin{pmatrix} 0 & \lambda^2 - 2\nu - 1 \\ 1 & 0 \end{pmatrix}. \tag{55}$$

The gauge transformation (17) yields the Lax pair of the following form:

$$\Psi_\lambda \Psi^{-1} = A(\lambda, x) = \begin{pmatrix} a\lambda + u & \lambda^2 + b\lambda + v \\ w & -a\lambda - u \end{pmatrix} + \frac{1}{\lambda - x}\begin{pmatrix} p & q \\ r & -p \end{pmatrix} \tag{56}$$

$$\Psi_x \Psi^{-1} = U(\lambda, x) = -\frac{1}{\lambda - x}\begin{pmatrix} p & q \\ r & -p \end{pmatrix} \tag{57}$$

with the compatibility condition

$$\begin{cases} a_x = r \\ b_x = -2p \\ u_x = (b + x)r \\ v_x = -2p(b + x) + 2aq \\ w_x = -2ar \\ p_x = xr(b + x) + vr - wq \\ q_x = -2xp(b + x) + 2xaq + 2uq - 2pv \\ r_x = -2xar - 2ur + 2pw \end{cases} \tag{58}$$

which allows the integrals

$$\begin{cases} a^2 + w = 1 \\ 2au + r + bw = 0 \\ 2ap + u^2 + (b + x)r + vw - a = \rho \qquad \rho = \text{ constant} \\ p^2 + qr = \gamma^2 \qquad \gamma = \text{ constant.} \end{cases} \tag{59}$$

As before, we chose the concrete values of the constants $a^2 + w$ and $2au + r + bw$ without loss of generality because these parameters are related to the affine transformations of the $\mathbb{C}^2$ space of pairs (λ, x); so putting these constants to unity and zero, respectively, we do nothing, but fix a certain origin and scales.

In contrast, the parameters ρ and γ describe formal monodromy of the Ψ-function at infinity and at the point $\lambda = x$, respectively.

Using the integrals (59), we reduce the system (58) to the system of four first-order equations for the functions a, b, p and r, and observe that the ratio $R = -2r/w$ satisfies the PIV equation [3]

$$R = -\frac{2r}{w}: \qquad R_{xx} = \frac{R_x^2}{2R} + \frac{3}{2}R^3 + 4xR^2 + 2(x^2 + \rho)R - \frac{8\gamma^2}{R} \tag{60}$$

with the parameters

$$\alpha = -\rho \qquad \beta = 16\gamma^2$$

while the functions a and b satisfy the first-order ODE and become auxiliary functions

$$\frac{a_x}{a^2-1} = \frac{1}{2}R \qquad b_x = \frac{a^2-1}{2a}Rb + \frac{1}{2}R_x + xaR + \frac{a^2+1}{4a}R^2.$$

The rest of the coefficients are determined uniquely by the expressions

$$u = \frac{1-a^2}{4a}(R-2b) \qquad w = 1-a^2 \qquad p = -\frac{1}{4}R_x - \frac{1}{2}(xa+u)R - \frac{1}{4}aR^2$$

$$q = 2\frac{p^2-\gamma^2}{(1-a^2)R} \qquad v = \frac{1}{1-a^2}(a+\rho-2ap-u^2) + \frac{1}{2}(b+x)R.$$

4.3. Gauge equivalence of the Lax pairs for PIV

4.3.1. Gauge equivalence of the Garnier system and (43) and (44). Jimbo and Miwa [8] presented the Lax pair originated in the work of Garnier [6], which, after proper change of notation and the shift of λ in x, can be written in the following form:

$$\frac{\partial\Psi^{\mathrm{JM}}}{\partial\lambda} = \mathcal{A}\Psi^{\mathrm{JM}} \qquad \frac{\partial\Psi^{\mathrm{JM}}}{\partial x} = \mathcal{U}\Psi^{\mathrm{JM}} \tag{61}$$

with the connection coefficients $\mathcal{A}$ and $\mathcal{U}$ defined as follows:

$$\mathcal{A} = \begin{pmatrix} \lambda & u \\ \frac{2}{u}(z-\tau) & -\lambda \end{pmatrix} + \frac{1}{\lambda - x}\begin{pmatrix} -z & -\frac{uy}{2} \\ \frac{2(z^2-\theta^2)}{uy} & z \end{pmatrix} \tag{62}$$

$$\mathcal{U} = -x\sigma_3 - \frac{1}{\lambda - x}\begin{pmatrix} -z & -\frac{uy}{2} \\ \frac{2(z^2-\theta^2)}{uy} & z \end{pmatrix}. \tag{63}$$

The matrix $\mathcal{A}$ (62) with proper change of notation coincides with the matrix A (43), but the matrix $\mathcal{U}$ (63) contains an extra term $-x\sigma_3$ in comparison with U (44). This term can be eliminated by the diagonal gauge transformation that is independent of λ, i.e.

$$\Psi(\lambda, x) = \mathrm{e}^{x^2\sigma_3/2}\Psi^{\mathrm{JM}}(\lambda, x). \tag{64}$$

4.3.2. Gauge equivalence of an alternative Lax pair and (43) and (44). Another Lax pair for PIV found by Kitaev [23] and later by Milne *et al* [24],

$$\frac{\partial\Psi^{\mathrm{K}}}{\partial\lambda} = \mathcal{A}\Psi^{\mathrm{K}} \qquad \frac{\partial\Psi^{\mathrm{K}}}{\partial x} = \mathcal{U}\Psi^{\mathrm{K}} \tag{65}$$

$$\mathcal{A} = \begin{pmatrix} \frac{1}{2}\lambda^3 + \lambda(x+uv) + \frac{\alpha}{\lambda} & \mathrm{i}(\lambda^2 u + 2xu + u') \\ \mathrm{i}(\lambda^2 v + 2xv + v') & -\frac{1}{2}\lambda^3 + \lambda(x+uv) + \frac{\alpha}{\lambda} \end{pmatrix} \tag{66}$$

$$\mathcal{U} = \begin{pmatrix} \frac{1}{2}\lambda^2 + uv & \mathrm{i}\lambda u \\ \mathrm{i}\lambda v & -\frac{1}{2}\lambda^2 - uv \end{pmatrix} \tag{67}$$

comes from the Lax pair of Kaup and Newell [25] for the modified nonlinear Schrödinger (MNS) equation by the use of similarity reduction of the MNS equation to the PIV equation of Ablowitz *et al* [26]. This pair is also gauge equivalent to the Lax pair (61)–(63). Indeed, if $\Psi^{\mathrm{K}}(\lambda, x)$ satisfied (65)–(67), then

$$\Psi(\lambda, x) = \mathrm{e}^{x^2\sigma_3/2}\begin{pmatrix} 1 & \mathrm{i}u \\ 0 & 1 \end{pmatrix}(2(\lambda - x))^{\sigma_3/8}\Psi^{\mathrm{K}}(\sqrt{2(\lambda - x)}, x) \tag{68}$$

satisfies our first-degree Lax pair (43) and (44) with

$$a = 0 \qquad b = \mathrm{i}\mathrm{e}^{x^2}(u' - u^2 v) \qquad c = \mathrm{i}\mathrm{e}^{-x^2}v \qquad p = \frac{\alpha}{2} + \frac{1}{4} - xuv + \frac{uv'}{2}$$

$$q = -\mathrm{i}u\,\mathrm{e}^{x^2}\left(\alpha + \frac{1}{2} - xuv + \frac{uv'}{2}\right) \qquad r = \frac{\mathrm{e}^{-x^2}}{\mathrm{i}u}\left(-xuv + \frac{uv'}{2}\right).$$

4.3.3. Gauge equivalence of (43) and (44), and (56) and (57). The Lax pairs (43) and (44), and (56) and (57) are also gauge equivalent to each other. Indeed, if $\Psi(\lambda)$ solves the system (56) and (57), then the function

$$\tilde{\Psi}(\lambda) = \begin{pmatrix} 1 & \lambda s + t \\ 0 & 1 \end{pmatrix} \Psi(\lambda) \qquad s = \frac{1}{a+1} \qquad t = 2\frac{ab(a+1)+u}{(a-2)(a+1)^2} \tag{69}$$

satisfies the system (43) and (44) with the parameters

$$\tilde{a} = u + sr + tw \qquad \tilde{b} = v + s - 2sp - 2tu - xs^2r - 2str - t^2w \qquad \tilde{c} = w$$
$$\tilde{p} = p + sx + tr \qquad \tilde{q} = q - 2spx - 2tp - x^2s^2r - 2xstr - t^2r \qquad \tilde{r} = r$$

so that the Painlevé functions $R = -2r/w$ (60) and $\tilde{R} = -2\tilde{r}/\tilde{c}$ (53) coincide with each other. For the integral operator (2), the transformation (69) yields its regular perturbation $K \mapsto K + R$ with the regular rank 1 integral operator R with the kernel $R(\lambda, \mu) = f(\lambda)g(\mu)$, where f and g are some parabolic cylinder functions.

5. Hypergeometric and confluent hypergeometric kernels

In this short section we note that the Garnier systems [6, 8] for PV and PVI, i.e.

$$\Psi_\lambda \Psi^{-1} = A_0(\lambda) + \frac{B}{\lambda - x} \qquad \Psi_x \Psi^{-1} = -\frac{B}{\lambda - x} \tag{70}$$

with the matrix

$$A_0(\lambda) = \frac{1}{2}\sigma_3 + \frac{B_0}{\lambda} \tag{71}$$

in the case of PV, and

$$A_0(\lambda) = \frac{B_0}{\lambda} + \frac{B_1}{\lambda - 1} \tag{72}$$

in the case of PVI, have the structure described by (20) and (21). Furthermore, the corresponding 'vacuum' equations (13)

$$\Phi_\lambda \Phi^{-1} = A_0(\lambda)$$

are solvable in terms of the Whittaker functions for (71) and in terms of hypergeometric functions for (72). Thus there is no surprise that the Laguerre and Jacobi polynomial kernels introduced by Tracy and Widom in [12] are related to PV and PVI.

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